

On interchanging the states of a pair of qudits

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Abstract—The qubit SWAP gate has been shown to be an integral component of quantum circuitry design. It permutes the states of two qubits and allows for the storage quantum information, teleportation of atomic or ionic states, and is a fundamental element in the circuit implementation of Shor’s algorithm. We consider the problem of generalising the SWAP gate beyond the qubit setting. We show that quantum circuit architectures completely described by instances of the CNOT gate can not implement a transposition of a pair of qudits for dimensions $d \equiv 3 \pmod{4}$. This is of interest to the question of construction a generalised quantum SWAP gate. The task of constructing generalised SWAP gates based on transpositions of qudit states is argued in terms of the signature of a permutation.

I. INTRODUCTION

The crux of successful quantum computation is the implementation of multiple quantum gates. The most elementary of multiple quantum gates is to consider some unitary operator U within a controlled- U two qubit operation. The corresponding transformation given by transformation is written as $|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U$ where the I operation represents the identity transformation. This controlled two qubit operator is so called since the application of U on the second qubit is decided by the state of the first qubit. The classic controlled- U gate is the controlled-NOT (CNOT) gate and its action with respect to the computational basis is given as $|x\rangle|y\rangle \mapsto |x\rangle|y \oplus x\rangle$ where \oplus represents addition modulo 2. The CNOT gate plays an important role in quantum computation (DiVincenzo (1998)). It is the quantum mechanical analogue of the classical connective XOR gate and is a principle component for universal computations. It can be used to produce maximally entangled states similar to the set of EPR pairs (Nielsen and Chuang (2000)). Furthermore, the controlled-NOT gate acts as a measurement gate (Deutsch (1989)) and provides a basis for a so-called nondemolition measurement (Chuang and Yamamoto (1996)) that permits the construction of a syndrome table as used in error detection and correction.

The quantum network approach to computation resembles the classical procedure to computing (Vlasov (2003)) where quantum circuits are formed from a composition of quantum states, quantum gates and quantum wires (Nielsen and Chuang (2000)). Computations are described within the Hilbert space $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$ of n qubits where each horizontal quantum circuit wire corresponds to the individual \mathbb{C}^2 subspaces. Vertical wires in a quantum circuit represent the *coupling* of arbitrary pairs of quantum gates in a manner similar to a controlled- U

gate. The *depth* of a circuit refers to the maximum number of gates required to effect necessary state changes. The *width* of a circuit is the maximum number of gates in operation in any one time frame. Quantum computations are then a finite sequence of quantum gates set along the quantum wires to effect suitable transformations. Unfortunately, there are only a handful of quantum gates that can be experimentally realised within the coherence time of their systems (Vatan and Williams (2004)). Those gates that have been experimentally demonstrated are said to be elements of the quantum gate library. Barenco *et al.* (1995) showed that any quantum operation on a set of n -qubits can be restricted to a composition of CNOT, and single qubit gates. For this reason, we say that the qubit gate library consisting of single qubit gates and CNOT is universal. Furthermore, it has become standard in quantum information to express any n -qubit quantum operation as a composition of single qubit gates and CNOT gates. Consequently, the CNOT gate has acquired special status as the hallmark of multiqubit control (Vidal and Dawson (2004)).

Researchers in universal circuit constructions have done considerable work optimising their constructions (Nielsen (2005)). In particular, Vatan and Williams (2004) construct a quantum circuit for a general two-qubit operation that requires at most three CNOT gates and fifteen one-qubit gates and show that their construction is optimal. Crucial to this result is the demand that the quantum circuit for the two-qubit SWAP gate requires at least three CNOT gates. Fig. 1 illustrates a quantum circuit swapping the states of two qubits; system \mathcal{A} begins in the state $|\psi\rangle$ and ends in the state $|\phi\rangle$ while system \mathcal{B} begins in the state $|\phi\rangle$ and ends in the state $|\psi\rangle$. The SWAP gate has become an integral feature of the circuitry design of the quantum Fourier transform where it can be used to store quantum information, to teleport atomic or ionic states (Liang and Li (2005)). It is also a fundamental element in the circuit implementation of Shor’s algorithm (Fowler *et al.* (2004)). More recently, a scheme to realise the quantum SWAP gate between flying and stationary qubits has been presented by Liang and Li (2005) where maintained that experimentally realising the quantum SWAP gate is a necessary condition for the networkability of quantum computation.

Most often it is assumed that a quantum computer is predicated on a collection of qubits. However, there has been the view to generalise to d -level, or *qudit*, quantum mechanical systems. In the context of information processing, it may be argued that there are advantages in moving from the qubit

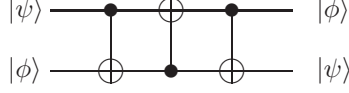


Fig. 1. Quantum circuit swapping two qubits.

paradigm to the qudit paradigm. For instance, as the entropy of a message depends on the alphabet used it ought to be that increasing the alphabet size should allow for the construction of better error-correcting codes (Grassl *et al.* (2003)). It has also been pointed out that a quantum system composed of a pair of three dimensional subsystems shows new features when compared to a two-qubit system (Grassl *et al.* (2003)).

We seek to establish conditions for generalising the quantum SWAP gate resulting through instances of the CNOT gate. We give the following results.

II. PRELIMINARIES

Consider the set $N = \{1, 2, \dots, n\}$ and let $\sigma : N \mapsto N$ be a bijection. We say $\sigma = \begin{bmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{bmatrix}$, where $i_k \in N$ is the image of $k \in N$ under σ , is a *permutation* of the set N .

Let σ and τ be two permutations of N . We define the product $\sigma \cdot \tau$ by $(\sigma \cdot \tau)(i) = \sigma(\tau(i))$, for $i \in N$, to be the composition of the mapping τ followed by σ . These permutations taken with (\cdot) form a group denoted S_n which is called the *symmetric group* of degree n .

Given the permutation σ and for each $i \in N$, let us consider the sequence $i, \sigma(i), \sigma^2(i), \dots$. Since σ is a bijection and N is finite there exist a smallest positive integer $\ell = \ell(i)$ depending on i such that $\sigma^\ell(i) = i$. The *orbit* of i under σ then consists of the elements $i, \sigma(i), \dots, \sigma^{\ell-1}(i)$. By a *cycle* of σ , we mean the ordered set $(i, \sigma(i), \dots, \sigma^{\ell-1}(i))$ which sends i into $\sigma(i)$, $\sigma(i)$ into $\sigma^2(i)$, \dots , $\sigma^{\ell-2}(i)$ into $\sigma^{\ell-1}(i)$, and $\sigma^{\ell-1}(i)$ into i and leaves all other elements of N fixed. Such a cycle is called an (ℓ) -cycle. We refer to 2-cycles as *transpositions*. A pair of elements $\{\sigma(i), \sigma(j)\}$ is an *inversion* in a permutation σ if $i < j$ and $\sigma(i) > \sigma(j)$. Any permutation can be written as a product of transpositions. The number of transpositions in any such product is even if and only if the number of inversions is even, and consequently, we say the permutation is even. Similarly, a permutation is odd if it can be written as a product of an odd number of transposition and hence has an odd number of inversions.

Lemma 1: Every permutation can be uniquely expressed as a product of disjoint cycles.

Proof: Let σ be a permutation. Then the cycles of the permutation are of the form $i, \sigma(i), \dots, \sigma^{\ell-1}(i)$. Since the cycles are disjoint and by the multiplication of cycles, we have it that the image of $i \in N$ under σ is the same as the image under the product, ς , of all the disjoint cycles of σ . Then, σ and ς have the same effect on every element in N , hence, $\sigma = \varsigma$.

Every permutation in S_n has then a *cycle decomposition* that is unique up to ordering of the cycles and up to a cyclic

permutation of the elements within each cycle. Further, if $\sigma \in S_n$ and σ is written as the product of disjoint cycles of length n_1, \dots, n_k , with $n_i \leq n_{i+1}$, we say (n_1, \dots, n_k) is the *cycle type* of σ .

As a result of Lemma 1, every permutation can be written as a product of transpositions. Since the number of transpositions needed to represent a given permutation is either even or odd, we define the *signature* of a permutation as

$$\text{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases} \quad (1)$$

To each permutation, let us associate a permutation matrix A_σ whereby

$$A_\sigma(j, i) = \begin{cases} 1 & \text{if } \sigma(i) = j \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

The mapping $f : S_n \mapsto \det(A_\sigma)$ is a group homomorphism, where

$$\det(A_\sigma) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n A_{\sigma(i), i} \quad (3)$$

The kernel of this homomorphism, $\ker f$, is the set of even permutations. Consequently, we have it that σ is even if and only if $\det(A_\sigma)$ equals $+1$. The kernel of the homomorphism signature defines the alternating group. Note that the set of odd permutation can not form a subgroup but they form a coset of the alternating group.

Let us consider the following problem. Given a pair of d -dimensional quantum systems, system \mathcal{A} in the state $|\psi\rangle$ and system \mathcal{B} in the state $|\phi\rangle$, determine if it is possible swap the states of the corresponding systems so that system \mathcal{A} is in the state $|\phi\rangle$ and that system \mathcal{B} is in the state $|\psi\rangle$.

III. INTERCHANGING A PAIR OF QUTRITS

Let $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$ be two d -dimensional Hilbert spaces with bases $|i\rangle_\mathcal{A}$ and $|i\rangle_\mathcal{B}$, $i \in \mathbb{Z}_d$ respectively. Let $|\psi\rangle_\mathcal{A}$ denote a pure state of the quantum system $\mathcal{H}_\mathcal{A}$. Similarly, let $|\phi\rangle_\mathcal{B}$ denote a pure state of the quantum system $\mathcal{H}_\mathcal{B}$ and consider an arbitrary unitary transformation $U \in U(d^2)$ acting on $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$. Let U_{CNOT1} (Vatan and Williams (2004)) denote a CNOT gate that has qudit $|\psi\rangle_\mathcal{A}$ as the control qudit and $|\phi\rangle_\mathcal{B}$ as the target qudit;

$$U_{\text{CNOT1}} |m\rangle_\mathcal{A} \otimes |n\rangle_\mathcal{B} = |m\rangle_\mathcal{A} \otimes |n \oplus m\rangle_\mathcal{B}, \quad m, n \in \mathbb{Z}_d \quad (4)$$

where $i \oplus j$ denote modulo d addition. In gate circuitry notation, the CNOT1 gate is given by

$$\begin{array}{c} |m\rangle_\mathcal{A} \text{---} \text{---} |m\rangle_\mathcal{A} \\ \quad \quad \quad \text{---} \oplus \text{---} |n \oplus m\rangle_\mathcal{B} \\ |n\rangle_\mathcal{B} \text{---} \text{---} |n\rangle_\mathcal{B} \end{array} \quad (5)$$

Similarly, let U_{CNOT2} denote a CNOT gate that has qudit $|\psi\rangle_\mathcal{A}$ as the target qudit and $|\phi\rangle_\mathcal{B}$ as the control qudit;

$$U_{\text{CNOT2}} |m\rangle_\mathcal{A} \otimes |n\rangle_\mathcal{B} = |m \oplus n\rangle_\mathcal{A} \otimes |n\rangle_\mathcal{B}, \quad m, n \in \mathbb{Z}_d \quad (6)$$

In gate circuitry notation, the CNOT2 gate is given by

$$\begin{array}{c} |m\rangle_{\mathcal{A}} \text{---} \oplus \text{---} |m \oplus n\rangle_{\mathcal{A}} \\ |n\rangle_{\mathcal{B}} \text{---} \bullet \text{---} |n\rangle_{\mathcal{B}} \end{array} \quad (7)$$

We now show that a swap of two qutrits is not possible using a composition of CNOT gates alone. The point of this argument is to illustrate that a quantum gate construction which permutes the states of three qutrit systems can not be described by a set of qutrit transpositions induced by the CNOT gate alone. Were this otherwise then a simple solution to the problem of construction a generalised SWAP gate for three qutrits. To argue this point, we first note that any sequence of CNOT gates acting on the qutrit states $|\psi\rangle_{\mathcal{A}}$ and $|\phi\rangle_{\mathcal{B}}$ can be written as a composition of the gates CNOT1 and CNOT2. The CNOT1 and CNOT2 gates can be described in the following way; the permutation matrix corresponding to the CNOT1 gate takes the value 1 in row $3m+n$ and column $3m+(m \oplus n)$, $m, n = 0, 1, 2$. Similarly, the matrix corresponding to the CNOT2 gate takes the value 1 in row $3m+n$ column $3(m \ominus n) + n$. These unitary matrix representations for a CNOT gate are given in Fig. 2. Furthermore, both the CNOT1 matrix and CNOT2 matrix have determinant +1 since the permutation corresponding to each of the respective matrices is even.

Let us now assume that there exists a gate that swaps a pair of qutrit states and that such a gate is composed using only the CNOT gate. Such a swap gate will then be a composition of the gates CNOT1 and CNOT2. Since each CNOT circuit acting on a pair of qutrits is a composition of CNOT1 and CNOT2, it follows that any such composition will be equivalent to some product of their respective unitary matrices. Such a product matrix product will necessarily have determinant +1 as its constituent elements have determinant +1. However, the matrix transformation representation required to effectuate the swap of a pair of qutrits is given in Fig. 3, and takes the value 1 in row $3m+n$ column $3n+m$ and has determinant -1. Thus, no composition of the former can yield the latter and the result follows.

IV. INTERCHANGING A PAIR OF QUDITS

Barenco *et al.* (1995) showed that any unitary transformation on a set of qubits can be decomposed into a sequence of CNOT and single-qubit gates (Vidal and Dawson (2004)). We now consider the problem of swapping a pair of d -dimensional quantum states using only CNOT gates such that the system $\mathcal{H}_{\mathcal{A}}$ begins in the state $|\psi\rangle_{\mathcal{A}}$ and ends in the state $|\phi\rangle_{\mathcal{A}}$ while correspondingly the system $\mathcal{H}_{\mathcal{B}}$ begins in the state $|\phi\rangle_{\mathcal{B}}$ and ends in the state $|\psi\rangle_{\mathcal{B}}$. Our argument will be that a transposition of qudit states induces some unitary matrix $U(d^2)$ over $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$ whose circuit architecture can not be completely determined by using only CNOT gates.

Recall the particular problem concerning the swap of a pair of qutrit systems. We have shown how the unitary matrices U_{CNOT1} and U_{CNOT2} both have determinant +1. We also showed that this is in contrast to matrix U_{SWAP} which describes the swapping of states of a pair of quantum systems where such a matrix has determinant -1. Consequently, no composition of

$$\begin{array}{c} \oplus \\ \bullet \end{array} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{c} \bullet \\ \oplus \end{array} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Fig. 2. Matrix representations of CNOT types.

$$U_{\text{SWAP}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Fig. 3. U_{SWAP}

CNOT gates alone can induce the matrix that determines the action of the SWAP gate. Another way to look at this is the following. The permutations

$$\begin{aligned} \sigma_{\text{CNOT1}} &= \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & 2 & 4 & 5 & 3 & 8 & 6 & 7 \end{pmatrix} \\ \sigma_{\text{SWAP}} &= \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 5 & 6 & 7 & 4 & 1 & 2 & 3 & 8 \end{pmatrix} \end{aligned} \quad (8)$$

that correspond to the unitary matrices U_{CNOT1} and U_{SWAP} have corresponding cycle types $(1, 1, 1, 3, 3)$ and $(1, 1, 1, 2, 2, 2)$. Hence, a CNOT gate fixes three basis states and permutes the remaining states in two cycles of length 3. Each such cycle may be written as a product of two transpositions. Whence, the signature of the CNOT permutation is +1. On the other hand, a SWAP gate that swaps the states of a pairs of qutrits contains three fixed elements and a set of three transpositions and therefore the signature of the SWAP permutation is -1 and it follows that no composition of CNOT gates can lead to an execution of a swap of a pair of qutrit systems.

More generally, a CNOT gate acting on a pair of d -dimensional quantum systems corresponds to a permutation of the d^2 basis states. We consider the case when $d = p$ is a prime. For prime dimensions $d = p$ and taking the case of CNOT1, we have it that the basis states $|m\rangle_{\mathcal{A}} \otimes |n\rangle_{\mathcal{B}}$ of the system \mathcal{H}_{AB} are mapped mapped to $|m\rangle_{\mathcal{A}} \otimes |n \oplus m\rangle_{\mathcal{B}}$. The permutation associated with the CNOT1 mapping fixes d basis states and has $(d-1)$ cycles of length d , each of which may be written as a product of $d-1$ transpositions. CNOT1 yields a permutation that can then be composed of $(d-1)^2$ transpositions of qudit basis states. Similarly, the CNOT2 gate acting on a pair of qudit basis states maps $|m\rangle_{\mathcal{A}} \otimes |n\rangle_{\mathcal{B}}$ of \mathcal{H}_{AB} to $|m \oplus n\rangle_{\mathcal{A}} \otimes |n\rangle_{\mathcal{B}}$. There are d fixed basis elements under the CNOT2 mapping and $(d-1)$ cycles, each a product of $d-1$ transpositions. Therefore, the signature of the CNOT permutation is -1 for dimension $d = 2$ and $+1$ for odd prime dimensions. Now suppose a CNOT gate is acting on a pair of qudits within system \mathcal{H}_{d^d} . Further suppose that such an action is described by $U_{\text{CNOT}} \otimes I_{d^{d-2}}$. This matrix representation induces a permutation of $d^{(d-2)}$ copies of the d^2 basis elements targeted by the CNOT gate and it follows that the signature of corresponding permutation is -1 only for dimension $d = 2$.

Let us consider a SWAP gate that swaps that states of a pair of qudits. Such a gate corresponds to a permutation of the d^2 basis states of system \mathcal{H}_{d^2} which maps basis states $|m\rangle_{\mathcal{A}} \otimes |n\rangle_{\mathcal{B}}$ to basis states $|n\rangle_{\mathcal{A}} \otimes |m\rangle_{\mathcal{B}}$. Under this mapping there are d fixed basis elements and $d(d-1)/2$ transpositions which describe the interchanging of all remaining basis states. Thus, the signature of the permutation corresponding to the SWAP gate of a pair of qudits is -1 for dimensions $d \equiv 2$ or $3 \pmod{4}$ and $+1$ for dimensions $d \equiv 0$ or $1 \pmod{4}$. Thus when $d \equiv 3 \pmod{4}$ the SWAP cannot be realised the CNOT gates alone. Further consider a cycle of d quantum states that maps basis states $|u\rangle_{\mathcal{I}} \otimes |v\rangle_{\mathcal{J}} \otimes |w\rangle_{\mathcal{K}} \dots \otimes |z\rangle_{\mathcal{M}}$ to the basis states $|z\rangle_{\mathcal{I}} \otimes |u\rangle_{\mathcal{J}} \otimes |v\rangle_{\mathcal{K}} \dots \otimes |y\rangle_{\mathcal{M}}$. As above the cycle structure of this permutation depends on the factorisation of the dimension of the quantum system. Thus, for prime dimensions, the permutation corresponding to a cycle of d qudit states contains d fixed states and $(d^d - d)/d$ cycles of length d . Consequently, there are $(d^{(d-1)} - 1)(d-1)$ transpositions association with the cycle of d qudit systems. Over even dimension d , the permutation signature of such is -1 and $+1$ for odd dimension d .

The task of interchanging a pair of qudit states has been argued in terms of the signature of a permutation. Based on this argument, we have shown that a CNOT gate acting on a pair of qudits corresponds to a permutation whose signature is $+1$, for odd prime dimensions. A SWAP of pairs of qudit systems yields a permutation whose signature is -1 for dimensions $d \equiv 2$ or $3 \pmod{4}$ and $+1$ for dimensions $d \equiv 0$ or $1 \pmod{4}$. By this argument alone, circuit architectures completely described by instances of the CNOT gate can not be used to implement a SWAP of a pair of qudits for dimensions $d \equiv 3 \pmod{4}$.

V. CONCLUSION

We have shown that quantum circuit architectures completely described by instances of the CNOT gate can not implement a transposition of a pair of qudits for dimension $d \equiv 3 \pmod{4}$. This is of interest as constructing a SWAP gates for qutrits can not be implemented through a sequence of transpositions of qutrits if only CNOT gates are used. We ask the question can a generalised SWAP gate for higher dimensional quantum systems can be constructed entirely from instances of the CNOT gate.

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